WEIGHTED, CIRCULAR AND SEMI-ALGEBRAIC PROOFS

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Weighted Resolution, Circular Resolution and Sherali-Adams are equivalent proof systems. Nullstellensatz is equivalent to a restricted version of Weighted Resolution.

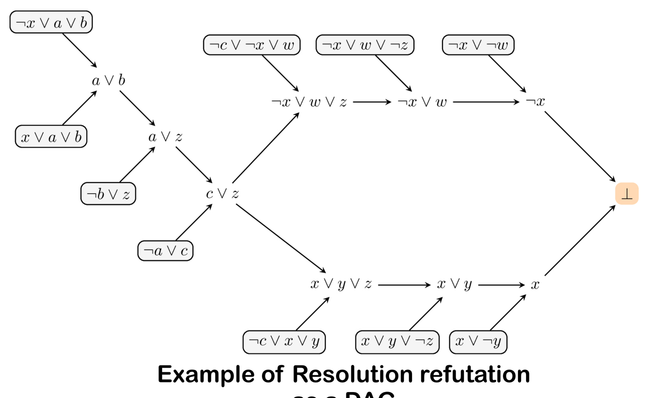
Intro

Proof complexity studies the complexity of proving true sentences in formal systems. The systems may capture different types of reasoning (logic, algebraic, geometric…).

Studying NP vs coNP is the original motivation. Now it is also relevant as a tool to show limitations of families of algorithms, e.g. CDCL SAT-solvers.

Resolution is a logic system to certify the unsatisfiability of set of clauses. It is at the base of state-of-the art SAT-solvers, and it uses the following inference rule:

$$\frac{A \vee x \quad B \vee \neg x}{A \vee B}$$



Example of Resolution refutation as a DAG

Given a set of clauses, MaxSAT is the problem of finding an assignment maximising the number of satisfied clauses among them. Proof systems for MaxSAT certify lower bounds on the number of falsified clauses.

The systems SA (aka cRes/wRes) in this poster can be used both in the context of SAT and MaxSAT and they are in a sweet spot: powerful and yet not so powerful as to prohibit the existence of an efficient proof search algorithm (under some restrictions, such as width or degree).

Semi-Algebraic Proofs

Sherali-Adams (SA) is a hierarchy of relaxations that aims to solve Integer Linear Programs. As a proof system, SA is a way to show that a set of polynomial inequalities implies another polynomial inequality, under Boolean solutions.

The polynomial inequalities $p_1 \geq 0, \ldots p_m \geq 0$ imply in SA $s \geq 0$ via an algebraic identity of the form:

$$s = q_0 + \sum_{i+1}^{m} q_i p_i$$

where the q_is only have positive coefficients, and we assume the p_is contain the polynomial $x^2 = x$ for all the variables to enforce Boolean solutions (and $x + \bar{x} = 1$ too for technical reasons)

$$\begin{aligned} &-\bar{x}\bar{y}\bar{z} - \bar{x}\bar{y}z - 2\bar{x}y - x \\ &\quad + \bar{x}\bar{y}(z + \bar{z} - 1) \bullet 0 \\ &\quad + 2\bar{x}(y + \bar{y} - 1) \bullet 1 \\ &\quad + (x + \bar{x} - 1) \bullet 2 \\ &\quad - (y + \bar{y} - 1) \bullet 3 \\ &\quad - \bar{y}(x + \bar{x} - 1) \bullet 4 \\ &\quad = -\bar{x} - y - x\bar{y} \end{aligned}$$

Coefficients in unary or binary give systems of different strengths.

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Weighted Resolution

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{(x \vee y \vee z, 1)\ (x \vee y \vee \neg z, 1)\ (x \vee \neg y, 2)\ (\neg x, 1)}{(x \vee \neg y, 2)\ (\neg x, 1)\ (x \vee y, 1)}\ \text{SYMM.CUT}\ \&\approx \bullet 0}{(\neg x, 1)\ (x, 2)\ (x \vee y, -1)}\ \text{SYMM.CUT}\ \&\approx \bullet 1}{(\bot, 1)\ (x, 1)\ (x \vee y, -1)}\ \text{SYMM.CUT}\ \&\approx \bullet 2}{(y, 1)\ (\neg y, 1)\ (x, 1)\ (x \vee y, -1)}\ \text{SPLIT}\ \&\approx \bullet 3}{(\neg y, 1)\ (\neg x \vee y, 1)\ (x, 1)}\ \text{SPLIT}\ \&\approx \bullet 4$$

$$\frac{(C \vee x, w) \quad (C \vee \neg x, w)}{(C, w)}\ \text{Symm.Cut}$$

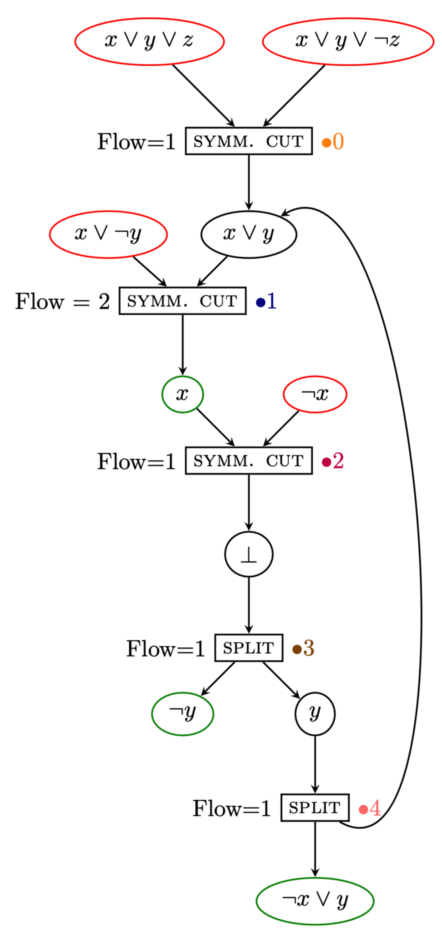
$$\frac{(C, w)}{(C \vee x, w) \quad (C \vee \neg x, w)}\ \text{Split}$$

Weighted Resolution (wRes) is a proof system for MaxSAT and it is a natural generalisation of MaxSAT-Resolution.

Given a multi-set of weighted clauses, wRes gives a way to certify a lower bound on the minimum weight of the clauses that are simultaneously falsified. wRes uses Symm.Cut, Split, and rules to combine weights ($\approx$) all as substitution rules. The goal is to derive a multiset with $(\bot, w)$ and all other clauses with positive weights.

Weights in $\mathbb{N}$ or $\mathbb{Z}$, in unary or binary give systems of different strengths.

Circular Resolution



Atserias & Lauria introduced the proof system Circular Resolution (cRes) as a way to strengthen Resolution proofs. In a cRes proof the directed graph of derived clauses allows cycles, as long as every clause is derived at least as many times as it is required as a premise.

There are two types of nodes (clauses and rules). The rules are Symmetric Cut and Split.

The soundness of the system is ensured by assigning to all rules a positive flow. The "balance" of the incoming flow minus the outgoing flow might be negative only in the premises and must be strictly positive in the conclusions.

cRes is equivalent to SA (doi:10.1145/357999)

No analogue of SA with unary coefficients or wRes with unary weights.

Equivalent systems

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